

ON THE FINE SPECTRUM OF THE FORWARD DIFFERENCE OPERATOR ON THE HAHN SPACE

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ABSTRACT. The main purpose of this paper is to determine the fine spectrum with respect to Goldberg's classification of the difference operator over the sequence space h . As a new development, we give the approximate point spectrum, defect spectrum and compression spectrum of the difference operator on the sequence space h .

1. INTRODUCTION

An important branch of mathematics due to its application in other branches of science is a Spectral Theory. It has been proved to be a very useful tool because of its convenient and easy applicability of the different fields. In numerical analysis, the spectral values may determine whether a discretization of a differential equation will get the right answer or how fast a conjugate gradient iteration will converge. In aeronautics, the spectral values may determine whether the flow over a wing is laminar or turbulent. In electrical engineering, it may determine the frequency response of an amplifier or the reliability of a power system. In quantum mechanics, it may determine atomic energy levels and thus, the frequency of a laser or the spectral signature of a star. In structural mechanics, it may determine whether an automobile is too noisy or whether a building will collapse in an earthquake. In ecology, the spectral values may determine whether a food web will settle into a steady equilibrium. In probability theory, they may determine the rate of convergence of a Markov process.

There are several studies about the spectrum of the linear operators defined by some triangle matrices over certain sequence spaces. This long-time behavior was intensively studied over many years, starting with the work by Wenger [29], who established the fine spectrum of the integer power of the Cesàro operator in c . The generalization of [29] to the weighted mean methods is due to Rhoades [26]. The study of the fine spectrum of the operator on the sequence space ℓ_p , ($1 < p < \infty$) was initiated by González [14]. The method of the spectrum of the Cesàro operator prepared by Reade [25], Akhmedov and Başar [1], and Okutoyi [21], respectively, whose established to this idea on the sequence spaces c_0 and bv . In [31], the fine spectrum of the Rhaly operators on the sequence spaces c_0 and c was given. The spectrum and fine spectrum for p-Cesàro operator acting on the space c_0 was studied by Coşkun [9]. The investigation of the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0 , c was made by Malafosse [20] and Altay and Başar [4], respectively, where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by $\|x\|_{s_r} = \sup_{k \in \mathbb{N}} |x_k|/r^k$, ($r > 0$). The idea

of the fine spectrum applied to the Zweier matrix which is a band matrix as an operator over the sequence spaces ℓ_1 and bv by Altay and Karakus [5]. Let Δ_ν is double sequential band matrix on ℓ_1 such that $(\Delta_\nu)_{nn} = \nu_n$ and $(\Delta_\nu)_{n+1,n} = -\nu_n$ for all $n \in \mathbb{N}$, under certain conditions on the sequence $\nu = (\nu_k)$. The spectra and the fine spectra of matrix Δ_ν were determined by Srivastava and Kumar [27]. Afterwards, these results of the double sequential band matrix Δ_ν generalized to the double sequential band matrix Δ_{uv} such that defined by $\Delta_{uv}x = (u_n x_n + v_{n-1} x_{n-1})_{n \in \mathbb{N}}$ for all $n \in \mathbb{N}$ (see [28]). In [6], the fine spectra of the Toeplitz operators represented by an upper and lower triangular n -band infinite matrices, over the sequence spaces c_0 and c was computed. The fine spectra of upper triangular double-band matrices over the sequence spaces c_0 and c was obtained by Karakaya and Altun [16]. Let $\Delta_{a,b}$ is a double band matrix with the convergent sequences $\tilde{a} = (a_k)$ and $\tilde{b} = (b_k)$ having certain properties, over the sequence space c . The fine spectrum of the matrix $\Delta_{a,b}$ was examined by Akhmedov and El-Shabrawy [3]. The approach to the fine spectrum with respect to Goldberg's classification studied with of the operator $B(r, s, t)$ defined by a triple band matrix over the sequence spaces ℓ_p and bv_p , ($1 < p < \infty$) by Furkan et al. [11]. Quite recently, the fine spectrum with respect to Goldberg's classification of the operator defined by the lambda matrix over the sequence spaces c_0 and c computed by Yeşilkayagil and Başar [30].

Hahn sequence space is defined as $x = (x_k)$ such that $\sum_{k=1}^{\infty} k|x_k - x_{k+1}|$ converges and x_k is a null sequence and is denoted by h . Initially, this space was defined and studied to some general properties by Hahn [15]. It was examined different properties of this space by Goes and Goes [12] and Rao [22], [23], [24]. Quite recently,

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the studies on Hahn sequence space has been compiled by Kirisci [17]. Also in [18], it has been defined a new Hahn sequence space by Cesàro mean.

In the present paper, our propose is to investigate the fine spectrum of the difference operator Δ on the sequence space h . And also, we define the approximate point spectrum, defect spectrum and compression spectrum of the difference operator on the sequence space h , as a new approach.

2. PRELIMINARIES AND DEFINITION

Let X and Y be Banach spaces, and also let $T : X \rightarrow Y$ be a bounded linear operator. The range of the operator T defined by

$$R(T) = \{y \in Y : y = Tx, x \in X\}.$$

The set of all bounded linear operators on X into itself denoted by $B(X)$.

We choose any Banach space X and let $T \in B(X)$. Then we can define the *adjoint* T^* of T is a bounded linear operator on the dual X^* of X such that $(T^*f)(x) = f(Tx)$ for all $f \in Y^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a non-trivial complex normed space. A linear operator T defined by $T : D(T) \rightarrow X$ on a subspace $D(T) \subseteq X$. We do not assume that $D(T)$ is dense in X or that T has closed graph $\{(x, Tx) : x \in D(T)\} \subseteq X \times X$. We mean by the expression "*T is invertible*" that there exists a bounded linear operator $S : R(T) \rightarrow X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. The perturbed operator defined on the same domain $D(T)$ as T as follows:

$$T_\alpha = \alpha I - T$$

such that associated with each complex number α . The *spectrum* $\sigma(T, X)$ consists of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T, X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

Let ω is the space of all complex valued sequences and ϕ the set of all infinitely nonzero sequences. A linear subspace of ω which contain ϕ said a *sequence space*. We write ℓ_∞ , c , c_0 and bv for the spaces of all bounded, convergent, null and bounded variation sequences which are the Banach spaces with the sup-norm $\|x\|_\infty = \sup_{k \in \mathbb{N}} |x_k|$ and $\|x\|_{bv} = \sum_{k=0}^{\infty} |x_k - x_{k+1}|$, respectively, while ϕ is not a Banach space with respect to any norm, where $\mathbb{N} = \{0, 1, 2, \dots\}$. Also by ℓ_p , we denote the space of all p -absolutely summable sequences which is a Banach space with the norm $\|x\|_p = \left(\sum_{k=0}^{\infty} |x_k|^p \right)^{1/p}$, where $1 \leq p < \infty$.

Let μ and ν be two sequence spaces, and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk} , where $k, n \in \mathbb{N}$. Then, we say that A defines a matrix mapping from μ into ν , and we denote it by writing $A : \mu \rightarrow \nu$ if for every sequence $x = (x_k) \in \mu$, the A -transform $Ax = \{(Ax)_n\}$ of x is in ν ; where

$$(2.1) \quad (Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \quad \text{for all } n \in \mathbb{N}.$$

By $(\mu : \nu)$, we denote the class of all matrices A such that $A : \mu \rightarrow \nu$. Thus, $A \in (\mu : \nu)$ if and only if the series on the right side of (2.1) converges for each $n \in \mathbb{N}$ and each $x \in \mu$, and we have $Ax = \{(Ax)_n\}_{n \in \mathbb{N}} \in \nu$ for all $x \in \mu$.

The BK -space h of all sequences $x = (x_k)$ such that

$$h = \left\{ x : \sum_{k=1}^{\infty} k |\Delta x_k| < \infty \quad \text{and} \quad \lim_{k \rightarrow \infty} x_k = 0 \right\}$$

was defined by Hahn [15]. Here and after Δ denotes the forward difference operator, that is, $\Delta x_k = x_k - x_{k+1}$, for all $k \in \mathbb{N}$. The following norm

$$\|x\|_h = \sum_k k |\Delta x_k| + \sup_k |x_k|$$

was given on the space h by Hahn [15] (and also [12]). Rao [22, Proposition 2.1] defined a new norm on h as $\|x\| = \sum_k k |\Delta x_k|$.

Hahn proved following properties of the space h :

Lemma 2.1. *The following statements hold:*

- (i) h is a Banach space.
- (ii) $h \subset \ell_1 \cap \int c_0$.
- (iii) $h^\beta = \rho_\infty$,

where $\int \lambda = \{x = (x_k) \in \omega : (kx_k) \in \lambda\}$ and $\rho_\infty = \{x = (x_k) \in \omega : \sup_n n^{-1} |\sum_{k=1}^n x_k| < \infty\}$.

Functional analytic properties of the BK -space $bv_0 \cap d\ell_1$ was studied by Goes and Goes [12], where $d\ell_1 = \{x = (x_k) \in \omega : \sum_{k=1}^\infty \frac{1}{k} |x_k| < \infty\}$. Also, in [12], the arithmetic means of sequences in bv_0 and $bv_0 \cap d\ell_1$ were considered, and used the fact that the Cesàro transform $(n^{-1} \sum_{k=1}^n x_k)$ of order one $x \in bv_0$ is a quasiconvex null sequence.

Rao [22] studied some geometric properties of Hahn sequence space and gave the characterizations of some classes of matrix transformations. Also, in [23] and [24], Rao examined the different properties of Hahn sequence space.

Balasubramanian and Pandiarani [8] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that β - and γ -duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences.

Until the new studies of Kirişçi [17, 18], there has not been any work containing the Hahn sequence space.

3. SUBDIVISION OF THE SPECTRUM

In this section, we define the parts called point spectrum, continuous spectrum, residual spectrum, approximate point spectrum, defect spectrum and compression spectrum of the spectrum. There are many different ways to subdivide the spectrum of a bounded linear operator. Some of them are motivated by applications to physics, in particular, quantum mechanics.

3.1. The Point Spectrum, Continuous Spectrum and Residual Spectrum. The name resolvent is appropriate since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1}y$ provided that T_α^{-1} exists. More importantly, the investigation of properties of T_α^{-1} will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α , and the spectral theory is concerned with those properties. For instance, we shall be interested in the set of all α 's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We shall also ask for what α 's the domain of T_α^{-1} is dense in X , to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and is bounded whose domain is dense in X . For our investigation of T , T_α and T_α^{-1} , we need some basic concepts in the spectral theory which are given, as follows (see Kreyszig [19, pp. 370-371]):

The *resolvent set* $\rho(T, X)$ of T is the set of all regular values α of T . Furthermore, the spectrum $\sigma(T, X)$ is partitioned into the following three disjoint sets:

The *point (discrete) spectrum* $\sigma_p(T, X)$ is the set such that T_α^{-1} does not exist. An $\alpha \in \sigma_p(T, X)$ is called an *eigenvalue* of T .

The *continuous spectrum* $\sigma_c(T, X)$ is the set such that T_α^{-1} exists and is unbounded, and the domain of T_α^{-1} is dense in X .

The *residual spectrum* $\sigma_r(T, X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X .

Therefore, these three subspectra form a disjoint subdivision such that

$$(3.1) \quad \sigma(T, X) = \sigma_p(T, X) \cup \sigma_c(T, X) \cup \sigma_r(T, X).$$

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we shall have to discuss. Indeed, it is well-known that $\sigma_c(T, X) = \sigma_r(T, X) = \emptyset$ and the spectrum $\sigma(T, X)$ consists of only the set $\sigma_p(T, X)$ in the finite-dimensional case.

3.2. The Approximate Point Spectrum, Defect Spectrum and Compression Spectrum. In this subsection, three more subdivision of the spectrum called the *approximate point spectrum*, *defect spectrum* and *compression spectrum* have been defined as in Appell et al. [7].

Let X is a Banach space and T is a bounded linear operator. A $(x_k) \in X$ *Weyl sequence* for T defined by $\|x_k\| = 1$ and $\|Tx_k\| \rightarrow 0$, as $k \rightarrow \infty$.

In what follows, we call the set

$$(3.2) \quad \sigma_{ap}(T, X) := \{\alpha \in \mathbb{C} : \text{there exists a Weyl sequence for } \alpha I - T\}$$

the *approximate point spectrum* of T . Moreover, the subspectrum

$$(3.3) \quad \sigma_\delta(T, X) := \{\alpha \in \mathbb{C} : \alpha I - T \text{ is not surjective}\}$$

is called *defect spectrum* of T .

The two subspectra given by (3.2) and (3.3) form a (not necessarily disjoint) subdivision

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_\delta(T, X)$$

of the spectrum. There is another subspectrum,

$$\sigma_{co}(T, X) = \{\alpha \in \mathbb{C} : \overline{R(\alpha I - T)} \neq X\}$$

which is often called *compression spectrum* in the literature. The compression spectrum gives rise to another (not necessarily disjoint) decomposition

$$\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_{co}(T, X)$$

of the spectrum. Clearly, $\sigma_p(T, X) \subseteq \sigma_{ap}(T, X)$ and $\sigma_{co}(T, X) \subseteq \sigma_\delta(T, X)$. Moreover, comparing these subspectra with those in (3.1) we note that

$$\begin{aligned}\sigma_r(T, X) &= \sigma_{co}(T, X) \setminus \sigma_p(T, X), \\ \sigma_c(T, X) &= \sigma(T, X) \setminus [\sigma_p(T, X) \cup \sigma_{co}(T, X)].\end{aligned}$$

Sometimes it is useful to relate the spectrum of a bounded linear operator to that of its adjoint. Building on classical existence and uniqueness results for linear operator equations in Banach spaces and their adjoints are also useful.

Proposition 3.1. [7, Proposition 1.3, p. 28] *The following relations on the spectrum and subspectrum of an operator $T \in B(X)$ and its adjoint $T^* \in B(X^*)$ hold:*

- (a) $\sigma(T^*, X^*) = \sigma(T, X)$.
- (b) $\sigma_c(T^*, X^*) \subseteq \sigma_{ap}(T, X)$.
- (c) $\sigma_{ap}(T^*, X^*) = \sigma_\delta(T, X)$.
- (d) $\sigma_\delta(T^*, X^*) = \sigma_{ap}(T, X)$.
- (e) $\sigma_p(T^*, X^*) = \sigma_{co}(T, X)$.
- (f) $\sigma_{co}(T^*, X^*) \supseteq \sigma_p(T, X)$.
- (g) $\sigma(T, X) = \sigma_{ap}(T, X) \cup \sigma_p(T^*, X^*) = \sigma_p(T, X) \cup \sigma_{ap}(T^*, X^*)$.

The relations (c)-(f) show that the approximate point spectrum is in a certain sense dual to the defect spectrum and the point spectrum is dual to the compression spectrum. The equality (g) implies, in particular, that $\sigma(T, X) = \sigma_{ap}(T, X)$ if X is a Hilbert space and T is normal. Roughly speaking, this shows that normal (in particular, self-adjoint) operators on Hilbert spaces are most similar to matrices in finite dimensional spaces (see Appell et al. [7]).

3.3. Goldberg's Classification of Spectrum. If X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$:

- (A) $R(T) = X$.
- (B) $R(T) \neq \overline{R(T)} = X$.
- (C) $\overline{R(T)} \neq X$.

and

- (1) T^{-1} exists and is continuous.
- (2) T^{-1} exists but is discontinuous.
- (3) T^{-1} does not exist.

If these possibilities are combined in all possible ways, nine different states are created. These are labelled by: $A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3$. If an operator is in state C_2 for example, then $\overline{R(T)} \neq X$ and T^{-1} exists but is discontinuous (see Goldberg [13]).

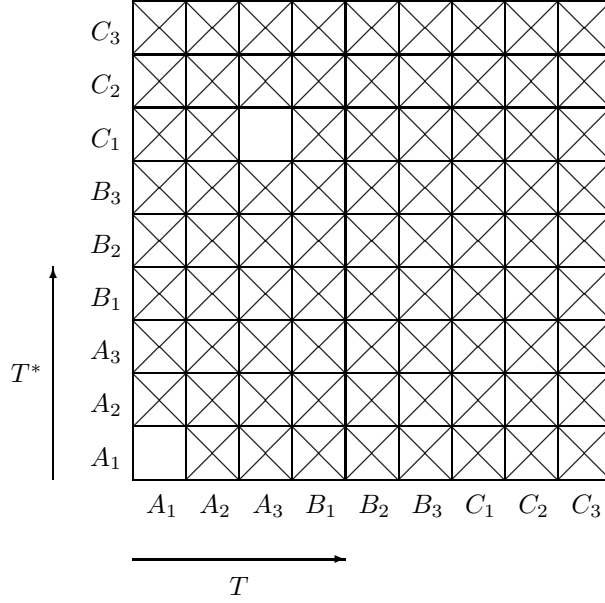


Table 1.1: State diagram for $B(X)$ and $B(X^*)$ for a non-reflective Banach space X

If α is a complex number such that $T_\alpha \in A_1$ or $T_\alpha \in B_1$, then $\alpha \in \rho(T, X)$. All scalar values of α not in $\rho(T, X)$ comprise the spectrum of T . The further classification of $\sigma(T, X)$ gives rise to the fine spectrum of T . That is, $\sigma(T, X)$ can be divided into the subsets $A_2\sigma(T, X) = \emptyset$, $A_3\sigma(T, X)$, $B_2\sigma(T, X)$, $B_3\sigma(T, X)$, $C_1\sigma(T, X)$, $C_2\sigma(T, X)$, $C_3\sigma(T, X)$. For example, if T_α is in a given state, C_2 (say), then we write $\alpha \in C_2\sigma(T, X)$.

By the definitions given above, we can illustrate the subdivision (3.1) in the following table:

		1	2	3
		T_α^{-1} exists and is bounded	T_α^{-1} exists and is unbounded	T_α^{-1} does not exist
A	$R(\alpha I - T) = X$	$\alpha \in \rho(T, X)$	–	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$
B	$\overline{R(\alpha I - T)} = X$	$\alpha \in \rho(T, X)$	$\alpha \in \sigma_c(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$
C	$\overline{R(\alpha I - T)} \neq X$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_r(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$	$\alpha \in \sigma_p(T, X)$ $\alpha \in \sigma_{ap}(T, X)$ $\alpha \in \sigma_\delta(T, X)$ $\alpha \in \sigma_{co}(T, X)$

Table 1.2: Subdivision of spectrum of a linear operator

One can observe by the closed graph theorem that in the case A_2 cannot occur in a Banach space X . If we are not in the third column of Table 1.2, i.e., if α is not an eigenvalue of T , we may always consider the resolvent operator T_α^{-1} (on a possibly *thin* domain of definition) as *algebraic inverse* of $\alpha I - T$.

The forward difference operator Δ is represented by the matrix

$$\Delta = \begin{bmatrix} 1 & -1 & 0 & 0 & \dots \\ 0 & 1 & -1 & 0 & \dots \\ 0 & 0 & 1 & -1 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Corollary 3.2. $\Delta : h \rightarrow h$ is a bounded linear operator.

4. ON THE FINE SPECTRUM OF THE FORWARD DIFFERENCE OPERATOR ON THE HAHN SPACE

In this section, we determine the spectrum and fine spectrum of the forward difference operator Δ on the Hahn space h and calculate the norm of the operator

Theorem 4.1. $\sigma(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| \leq 1\}$.

Proof. Let $|1 - \alpha| > 1$. Since $\Delta - \alpha I$ is triangle, $(\Delta - \alpha I)^{-1}$ exists and solving the matrix equation $(\Delta - \alpha I)x = y$ for x in terms of y gives the matrix $(\Delta - \alpha I)^{-1} = B = (b_{nk})$, where

$$b_{nk} = \begin{cases} \frac{1}{(1-\alpha)^{n+1}} & , \quad 0 \leq k \leq n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. Thus, we observe that

$$\begin{aligned} \|(\Delta - \alpha I)^{-1}\|_{(h:h)} &= \sum_{n=1}^{\infty} n |b_{nk} - b_{n+1,k}| \\ &\leq \sum_{n=1}^{\infty} n |b_{nk}| + \sum_{n=1}^{\infty} n |b_{n+1,k}| \\ &= \sum_{n=1}^{\infty} \frac{n}{|1-\alpha|^{n+1}} + \sum_{n=1}^{\infty} \frac{n}{|1-\alpha|^{n+2}}. \end{aligned}$$

From the ratio test, we have

$$\|(\Delta - \alpha I)^{-1}\|_{(h:h)} < \infty,$$

that is, $(\Delta - \alpha I)^{-1} \in (h : h)$. But for $|1 - \alpha| \leq 1$,

$$\|(\Delta - \alpha I)^{-1}\|_{(h:h)} = \infty,$$

that is, $(\Delta - \alpha I)^{-1}$ is not in $B(h)$. This completes the proof. \square

Theorem 4.2. $\sigma_p(\Delta, h) = \emptyset$.

Proof. Suppose that $\Delta x = \alpha x$ for $x \neq \theta$ in h . Then, by solving the system of linear equations

$$\begin{aligned} x_0 &= \alpha x_0, \\ -x_0 + x_1 &= \alpha x_1 \\ -x_1 + x_2 &= \alpha x_2 \\ &\vdots \\ -x_{n-1} + x_n &= \alpha x_n \\ &\vdots \end{aligned}$$

we find that if x_{n_0} is the first nonzero entry of the sequence $x = (x_n)$, then $\alpha = 1$. From the equality $-x_{n_0} + x_{n_0+1} = \alpha x_{n_0+1}$ we have x_{n_0} is zero. This contradicts the fact that $x_{n_0} \neq 0$, which completes the proof. \square

Theorem 4.3. $\sigma_p(\Delta^*, h^*) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$.

Proof. Suppose that $\Delta^* x = \alpha x$ for $x \neq \theta$ in $h^* \cong \sigma_\infty$. Then, by solving the system of linear equations

$$\begin{aligned} x_0 - x_1 &= \alpha x_0, \\ x_1 - x_2 &= \alpha x_1, \\ &\vdots \\ x_{n-1} - x_n &= \alpha x_n, \\ &\vdots \end{aligned}$$

we observe that $x_n = (1 - \alpha)^n x_0$. Therefore, $\sup_n \frac{|x_0|}{n} \sum_{k=1}^n |1 - \alpha|^k < \infty$ if and only if $|1 - \alpha| < 1$. This step concludes the proof. \square

If $T \in B(h)$ with the matrix A , then it is known that the adjoint operator $T^* : h^* \rightarrow h^*$ is defined by the transpose A^t of the matrix A . It should be noted that the dual space h^* of h is isometrically isomorphic to the Banach space σ_∞ of absolutely summable sequences normed by $\|x\| = \sum_{k=0}^{\infty} k|x_k - x_{k+1}|$.

Lemma 4.4. [13, p. 59] T has a dense range if and only if T^* is one to one.

Theorem 4.5. $\sigma_r(\Delta, h) = \sigma_p(\Delta^*, h^*)$.

Proof. For $|1 - \alpha| < 1$, the operator $\Delta - \alpha I$ is triangle, so has an inverse. But $\Delta^* - \alpha I$ is not one to one by Theorem 4.3. Therefore by Lemma 4.4, $\overline{R(\Delta - \alpha I)} \neq h$ and this step concludes the proof. \square

Theorem 4.6. $\sigma_c(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| = 1\}$.

Proof. For $|1 - \alpha| < 1$, the operator $\Delta - \alpha I$ is triangle, so has an inverse but is unbounded. Also $\Delta^* - \alpha I$ is one to one by Theorem 4.3. By Lemma 4.4, $\overline{R(\Delta - \alpha I)} = h$. Thus, the proof is completed. \square

Theorem 4.7. $A_3\sigma(\Delta, h) = B_3\sigma(\Delta, h) = C_3\sigma(\Delta, h) = \emptyset$.

Proof. From Theorem 4.2 and Table 1.2., $A_3\sigma(\Delta, h) = B_3\sigma(\Delta, h) = C_3\sigma(\Delta, h) = \emptyset$ is observed. \square

Theorem 4.8. $C_1\sigma(\Delta, h) = \emptyset$ and $\alpha \in \sigma_r(\Delta, h) \cap C_2\sigma(\Delta, h)$.

Proof. We know $C_1\sigma(\Delta, h) \cup C_2\sigma(\Delta, h) = \sigma_r(\Delta, h)$ from Table 1.2. For $\alpha \in \sigma_r(\Delta, h)$, the operator $(\Delta - \alpha I)^{-1}$ is unbounded by Theorem 4.1. So $C_1\sigma(\Delta, h) = \emptyset$. This completes the proof. \square

Theorem 4.9. The following results hold:

- (a) $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.
- (b) $\sigma_\delta(\Delta, h) = \sigma(\Delta, h)$.
- (c) $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$.

Proof. (a) Since $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h) \setminus C_1\sigma(\Delta, h)$ from Table 1.2. and $C_1\sigma(\Delta, h) = \emptyset$ by Theorem 4.8, we have $\sigma_{ap}(\Delta, h) = \sigma(\Delta, h)$.

(b) Since $\sigma_\delta(\Delta, h) = \sigma(\Delta, h) \setminus A_3\sigma(\Delta, h)$ from Table 1.2 and $A_3\sigma(\Delta, h) = \emptyset$ by Theorem 4.7, we have $\sigma_\delta(\Delta, h) = \sigma(\Delta, h)$.

(c) Since the equality $\sigma_{co}(\Delta, h) = C_1\sigma(\Delta, h) \cup C_2\sigma(\Delta, h) \cup C_3\sigma(\Delta, h)$ holds from Table 1.2, we have $\sigma_{co}(\Delta, h) = \{\alpha \in \mathbb{C} : |1 - \alpha| < 1\}$ by Theorems 4.8 and 4.9. \square

The next corollary can be obtained from Proposition 2.1.

Corollary 4.10. The following results hold:

- (a) $\sigma_{ap}(\Delta^*, \ell_1) = \sigma(\Delta, h)$.
- (b) $\sigma_\delta(\Delta^*, \ell_1) = \{\alpha : |\alpha - (2 - \delta)^{-1}| = (1 - \delta)(2 - \delta)\} \cup E$.
- (c) $\sigma_p(\Delta^*, \ell_1) = \{\alpha \in \mathbb{C} : |\alpha - (2 - \delta)^{-1}| < (1 - \delta)/(2 - \delta)\} \cup S$.

5. CONCLUSION

Hahn [15] defined the space h and gave its some general properties. Goes and Goes [12] studied the functional analytic properties of the space h . The study on the Hahn sequence space was initiated by Rao [22] with certain specific purpose in Banach space theory. Also Rao [22] emphasized on some matrix transformations. Rao and Srinivasalu [23] introduced a new class of sequence space called the semi replete space. Rao and Subramanian [24] defined the semi Hahn space and proved that the intersection of all semi Hahn spaces is Hahn space. Balasubramanian and Pandiarani [8] defined the new sequence space $h(F)$ called the Hahn sequence space of fuzzy numbers and proved that β - and γ -duals of $h(F)$ is the Cesàro space of the set of all fuzzy bounded sequences. The sequence space h was introduced by Hahn [15] and Goes and Goes [12], and Rao [22, 23, 24] investigated some properties of the space h . Quite recently, Kirişçi [17] has defined a new Hahn sequence space by using Cesàro mean, in [18].

The difference matrix Δ was used for determining the spectrum or fine spectrum acting as a linear operator on any of the classical sequence spaces c_0 and c , ℓ_1 and bv , ℓ_p for $(1 \leq p < \infty)$, respectively in [4], [10] and [2].

As a natural continuation of this paper, one can study the spectrum and fine spectrum of the Cesàro operator, Weighted mean operator or another known operators in the sequence space h .

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